On the non-linear mechanics of wave disturbances in stable and unstable parallel flows

Part 2. The development of a solution for plane Poiseuille flow and for plane Couette flow

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In Part 1 by Stuart (1960), a study was made of the growth of an unstable infinitesimal disturbance, or the decay of a finite disturbance through a stable infinitesimal disturbance to zero, in plane Poiseuille flow, and that paper gave the most important terms in a solution of the equations of motion. The greater part of the present paper is concerned with a re-formulation of this problem which readily yields the complete solution. By the same method a solution for Couette flow is obtained. This solution is only a formal one for the present because the conditions imposed in deriving the solution may not be valid for Couette flow; this flow is believed to be stable to infinitesimal disturbances of the type considered.

1. Introduction

The object of this paper is to extend Stuart's work in Part 1 (1960) to give the full solution and to apply a similar analysis to give a formal solution for plane Couette flow. The analysis presented is a modification and extension of that used by Stuart. Briefly the method used is to represent the stream function by a Fourier series in the co-ordinate parallel to the direction of flow and equate each component, giving an infinite set of partial differential equations in two variables. This infinite set of equations is then solved by assuming a separable solution, thereby reducing the set to one involving only ordinary differential equations, some of which are solved by the method of perturbations, using c_i as the small parameter. The most important concept introduced by Stuart and used here is that, in the separable solution mentioned above, an undetermined time amplitude function replaces the exponential function of linear theory and this amplitude function is chosen to make the series converge for a greater range in time than the range for which the linear theory may be said to be valid. In other words, by this choice of the time amplitude function the major terms in the solution will represent the behaviour of the disturbance for a greater range in time than does the linear disturbance.

2. Fourier analysis of governing equations

The Navier–Stokes and continuity equations for two-dimensional, incompressible flow between parallel planes may be written in the form

$$u_t + uu_x + wu_z = -p_x + \frac{1}{R}(u_{xx} + u_{zz}), \tag{2.1}$$

$$w_t + uw_x + ww_z = -p_z + \frac{1}{R}(w_{xx} + w_{zz}), \tag{2.2}$$

$$u_x + w_z = 0, \tag{2.3}$$

where x denotes the distance parallel to the planes, z the distance normal to them measured from the channel centre, u, w the corresponding velocity components, p the pressure, R the Reynolds number and t the time. The suffices indicate differentiation. All quantities have been made non-dimensional, the reference length being half the distance between the planes (h), the reference velocity being the maximum velocity in steady flow (U_0), the reference time being h/U_0, and the reference pressure being ρU_0^2, where ρ is the density of the fluid. The Reynolds number is $R = U_0 h/\nu$, where ν is the kinematic viscosity. The fundamental solutions for steady flow are as follows:

(i) For flow under a pressure gradient between fixed planes (plane Poiseuille flow),

$$u = 1 - z^2, \quad w = 0, \quad -p_x = \frac{2}{R}, \quad -p_z = 0. \tag{2.4}$$

(ii) For flow under no pressure gradient between the planes in constant relative motion (plane Couette flow),

$$u = z, \quad w = 0, \quad -p_x = 0, \quad -p_z = 0. \tag{2.5}$$

In case (i) the type of infinitesimal disturbance considered is one which is travelling in the direction of flow (in the positive x direction) having a stream function of the form

$$\psi = C\psi_1(z)\exp[i\alpha(x-ct)] + \tilde{C}\tilde{\psi}_1(z)\exp[-i\alpha(x-\tilde{c}t)], \tag{2.6}$$

where $c = c_r + ic_i$ ($c_r \geqslant 0$), the symbol $\sim$ denotes a complex conjugate, and C is an arbitrary constant.

In case (ii) the type of infinitesimal disturbance considered is composed of two disturbances travelling in opposite directions with stream function of the form

$$\psi = C\psi_1(z)\exp[i\alpha(x-ct)] + \tilde{C}\tilde{\psi}_1(z)\exp[-i\alpha(x-\tilde{c}t)] + \tilde{C}\tilde{\psi}_1(-z)\exp[i\alpha(x+\tilde{c}t)] + C\psi_1(-z)\exp[-i\alpha(x+ct)], \tag{2.7}$$

where C is an arbitrary constant. It is found that in turbulent Couette flow the mean flow is antisymmetric so that although a disturbance of the form (2.6) is possible, it would not in general lead to an antisymmetrical mean flow while (2.7) does.

The linear equation for ψ_1, together with the homogeneous boundary conditions corresponding to the vanishing of the velocity components on the planes, constitute an eigenvalue problem to determine c as a function of α and R. There will in general be a sequence of values of c and of corresponding eigenfunctions, ψ_1, for given α and R. In case (i) for given α, R in the supercritical region it appears that there is only one unstable infinitesimal disturbance, that is, there is only one eigenvalue c with c_i positive; furthermore, this corresponds to an eigenfunction which is an even function of z. Now in the non-linear theory the convergence of the solution obtained is expected to be most rapid when that c is chosen where c_i has the smallest magnitude. To this eigenvalue corresponds an eigenfunction and the associated infinitesimal disturbance is used as a basis in the non-linear theory. It might be expected that the desired eigenvalue c in subcritical Poiseuille flow corresponds to an eigenfunction which is even in z. Accordingly, for this flow, that infinitesimal disturbance is chosen which has the algebraically largest value of c_i for even eigenfunctions.

The stream functions representing infinitesimal disturbances in Poiseuille and Couette flow both involve the sum of terms of the form $f(z,t)\,e^{i\alpha x}$ and $F(z,t)\,e^{-i\alpha x}$. These stream functions satisfy the linear equations exactly but when the non-linearity of the equations is not neglected each disturbance reacts with itself and with the main flow, generating higher harmonics of the form

$$f_n(z,t)\,e^{ni\alpha x} \quad (n = \pm 2, \pm 3, \ldots).$$

It therefore appears permissible to expand the stream function of the flow when non-linearity is included as a Fourier series in x.

Let the stream function for the flow be represented as the Fourier series,

$$\psi(x,z,t) = \bar{\phi}+\phi' = \bar{\phi}(z,t) + \sum_{n=1}^{\infty} \{\phi_n(z,t)\,e^{ni\alpha x} + \tilde{\phi}_n(z,t)\,e^{-ni\alpha x}\}. \tag{2.8}$$

Then
$$u = \bar{u}+u' = \bar{u}(z,t) + \sum_{n=1}^{\infty} \{u'_n(z,t)\,e^{ni\alpha x} + \tilde{u}'_n(z,t)\,e^{-ni\alpha x}\}, \tag{2.9}$$

and
$$w = w' = \sum_{n=1}^{\infty} \{w'_n(z,t)\,e^{ni\alpha x} + \tilde{w}'_n(z,t)\,e^{-ni\alpha x}\}, \tag{2.10}$$

where $\bar{u}(z,t) \equiv \bar{\phi}_z,\quad u'_n(z,t) \equiv \phi_{nz},\quad w'_n(z,t) \equiv -ni\alpha\phi_n \quad (n \geqslant 1).$

In linearized theory $\bar{\phi}$ represents the steady stream function and the other part of the right-hand side of (2.8) reduces to (2.6) or (2.7), which represents the disturbance stream function. In the non-linear theory the sum on the right-hand side of (2.8) represents the finite disturbance while $\bar{\phi}$ is the mean stream function, where the mean is taken with respect to x over the wavelength of the disturbance, $2\pi/\alpha$. The expressions (2.9) and (2.10) are to be substituted into (2.1) and (2.2), and the Fourier components are to be equated. In order that the pressure gradients shall balance the remaining terms the pressure must be of the form

$$\begin{aligned} p &= xp^*(t) + p^{**}(z,t) + p'(x,z,t) \\ &= xp^*(t) + p^{**}(z,t) + \sum_{n=1}^{\infty} \{p_n(z,t)\,e^{ni\alpha x} + \tilde{p}_n(z,t)\,e^{-ni\alpha x}\}. \end{aligned} \tag{2.11}$$

The conditions to be applied are (i) that the mean velocity $\bar{u}$ assumes the same values on the walls as does the undisturbed velocity, (ii) that the disturbance velocities, u', w', vanish at the walls, and (iii) that a suitable condition on the mean pressure gradient in the flow direction or, equivalently on the mean velocity. Hence, for Poiseuille flow,

$$\overline{u},\quad u_n',\quad w_n' = 0 \quad \text{at} \quad z = \pm 1 \quad (n = 1, 2, \ldots),$$

or

$$\overline{u},\quad \phi_{nz},\quad \phi_n = 0 \quad \text{at} \quad z = \pm 1 \quad (n = 1, 2, \ldots); \tag{2.12}$$

and, for Couette flow,

$$\overline{u} = 1 \quad \text{at} \quad z = 1, \qquad \overline{u} = -1 \quad \text{at} \quad z = -1;$$

$$\phi_{nz} = \phi_n = 0 \quad \text{at} \quad z = \pm 1 \quad (n = 1, 2, \ldots). \tag{2.13}$$

Substitute (2.9), (2.10) and (2.11) into (2.1) and (2.2) and equate components. The equations arising from equating the terms independent of x are equivalently found by taking the mean of equations (2.1) and (2.2), and these are (Stuart 1956 a, b)

$$\overline{u}_t + \overline{u'u_x'} + \overline{w'u_z'} = -p^* + \frac{1}{R}\overline{u}_{zz}, \tag{2.14}$$

$$\overline{u'w_x'} + \overline{w'w_z'} = -p_z^{**}. \tag{2.15}$$

It is readily seen that these may be written in the form

$$\overline{u}_t + (\overline{u'w'})_z = -p^* + \frac{1}{R}\overline{u}_{zz}, \tag{2.16}$$

$$(\overline{w'^2})_z = -p_z^{**}. \tag{2.17}$$

The terms $(\overline{u'w'})_z$ and $(\overline{w'^2})_z$ are the Reynolds stress terms and they represent the effect of the disturbance on the mean motion; in the linear theory they are neglected. The disturbance equations are found on subtracting (2.14) from (2.1) and (2.15) from (2.2) and they are (Stuart 1956 a, b)

$$u_t' + \overline{u}u_x' + w'\overline{u}_z + \chi_1 = -p_x' + \frac{1}{R}(u_{xx}' + u_{zz}'), \tag{2.18}$$

$$w_t' + \overline{u}w_x' + \chi_2 = -p_z' + \frac{1}{R}(w_{xx}' + w_{zz}'), \tag{2.19}$$

where

$$\chi_1 = u'u_x' + w'u_z' - \overline{u'u_x'} - \overline{w'u_z'}, \tag{2.20}$$

$$\chi_2 = u'w_x' + w'w_z' - \overline{u'w_x'} - \overline{w'w_z'}. \tag{2.21}$$

The quantities χ_1 and χ_2 are the non-linear parts of the disturbance equations and they are neglected in the linear theory.

The equation of continuity will be automatically satisfied when use is made of the stream function (2.8). The dependent variables $\overline{u}$, ϕ_n, p' are determined from the three equations (2.16), (2.18) and (2.19); p^* is to be determined by condition (iii) above. Integration with respect to z of (2.17) then gives p^{**} to within an arbitrary additive function of time.

In terms of the stream function (2.8), equation (2.16) is

$$\overline{u}_t + i\alpha \sum_{n=1}^{\infty} n(\phi_{nz}\tilde{\phi}_n - \phi_n\tilde{\phi}_{nz})_z = -p^* + \frac{1}{R}\overline{u}_{zz}. \tag{2.22}$$

Eliminate p' between (2.18) and (2.19) by differentiating the former with respect to z, the latter with respect to x and substracting one from the other. The stream function (2.8) is inserted into the resulting equation and the nth ($n \geqslant 1$) component is selected, yielding

$$
\begin{aligned}
\mathscr{L}(n\alpha)\,\phi_n = \frac{1}{n}\Bigg[& \sum_{m=n+1}^{\infty} (m-n)\,\phi'_m\{\bar{\phi}''_{m-n} - (m-n)^2\alpha^2\bar{\phi}_{m-n}\} \\
& - \sum_{m=1}^{n-1} (n-m)\,\phi'_m\{\phi''_{n-m} - (n-m)^2\alpha^2\phi_{n-m}\} \\
& - \sum_{m=1}^{\infty} (n+m)\,\bar{\phi}'_m\{\phi''_{n+m} - (n+m)^2\alpha^2\phi_{n+m}\} \\
& - \sum_{m=1}^{\infty} m\bar{\phi}_m\{\phi'''_{n+m} - (n+m)^2\alpha^2\phi'_{n+m}\} \\
& + \sum_{m=1}^{n-1} m\phi_m\{\phi'''_{n-m} - (n-m)^2\alpha^2\phi'_{n-m}\} \\
& + \sum_{m=n+1}^{\infty} m\phi_m\{\bar{\phi}'''_{m-n} - (m-n)^2\alpha^2\bar{\phi}'_{m-n}\}\Bigg] \quad (n \geqslant 1), \qquad (2.23)
\end{aligned}
$$

where the operator $\mathscr{L}(n\alpha)$ is

$$
\mathscr{L}(n\alpha) \equiv \left(\bar{u} - \frac{i}{n\alpha}\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial z^2} - n^2\alpha^2\right) - \bar{u}'' + \frac{i}{n\alpha R}\left(\frac{\partial^2}{\partial z^2} - n^2\alpha^2\right)^2,
$$

where it is understood that the summations from $m = 1$ to $m = n-1$ are to be omitted when $n = 1$, and where a prime indicates differentiation with respect to z.

The next step is to determine ϕ_n and $\bar{u}$ from the infinite set of partial differential equations (2.22) and (2.23) in two variables. From this stage the analysis for Poiseuille and Couette flow differ, although the principles are the same. The difference is due to the difference in form of the infinitesimal disturbances (2.6) and (2.7) which form the basis of the subsequent analysis. The simpler case, that of Poiseuille flow, will be dealt with in § 2.1 and Couette flow in § 2.2.

2.1. *Solution for Poiseuille flow*

A solution of the equations of motion is sought which represents a small finite disturbance with time-dependent amplitude and with the property that, as the amplitude tends to zero, the disturbance tends through the infinitesimal disturbance (2.6) to zero (as $t \to \pm\infty$, according to whether the flow is subcritical or supercritical). Therefore, as the amplitude tends to zero, the disturbance stream function, ϕ', given in (2.8), must tend to the infinitesimal value (2.6), so that, on comparing components, $\phi_1 \sim C\psi_1(z)\,e^{-i\alpha ct}$, while $\phi_n \to 0$ ($n > 1$) more rapidly. It appears from this that a solution might be obtained in which ϕ_n ($n \geqslant 1$) is 'separable' and we look for such a solution. It is therefore expected that the highest-order term in ϕ_1, that which approaches $C\psi_1(z)\,e^{-i\alpha ct}$ as the amplitude tends to zero, will be of the form $A(t)\,\psi_1(z)$, where $A(t)$ is some, possibly bounded, function which behaves like $C\,e^{-i\alpha ct}$ as $A \to 0$. If there exist finite disturbances which are in equilibrium, then we expect to find a set of such disturbances such that as the neutral curve is approached the equilibrium amplitude tends to zero. In any case we look for a solution in which $|A|$ is small. Since $\phi_2 \to 0$ more rapidly than does ϕ_1 as $A \to 0$, then ϕ_2 must be of smaller order than $|A|$. In fact, in the right-hand side of (2.23) for $n = 2$, the terms depending on ϕ_1 only are of order ϕ_1^2 or $|A|^2$ and occur only in the form of the product of A^2

with a function of z. If it is assumed for the moment that the remaining terms are of lower order, then the main term in ϕ_2 is expected to be of the form $A^2\psi_2(z)$. From the fact that A is proportional to $e^{-i\alpha ct}$ as $A \to 0$, it follows that

$$\frac{1}{A}\frac{dA}{dt} \to -i\alpha c \quad \text{as} \quad A \to 0.$$

If then we look for a solution for which $(1/A)\,dA/dt$ is a function of A and $\tilde{A}$ only, then it follows that

$$\frac{dA}{dt} = -i\alpha c A + \text{smaller-order terms}.$$

To the highest order, the departure of $\bar{u}$ from the steady laminar value, $\bar{u}_l = 1-z^2$, is found from (2.22) retaining only the highest-order term in ϕ_1 in the sum and if this difference is to be separable then it is readily seen that it must be of the form

$$\bar{u}-\bar{u}_l = A\tilde{A}f_1(z) + \text{smaller-order terms}.$$

Returning now to (2.23) for $n = 1$, we have that the highest-order terms on the right-hand side are of the form of the product of $A^2\tilde{A}$ with a function of z (arising from terms like $\phi_2'\tilde{\phi}_1''$), provided that the remaining terms are of smaller order than this. The highest-order terms on the left-hand side being of order A must cancel out; that is, the coefficient of A in ϕ_1 will satisfy a certain differential equation. Dividing (2.23) with $n = 1$ by A and letting $A \to 0$, we see that this differential equation is

$$(\bar{u}_l-c)(\psi_1''-\alpha^2\psi_1)-\bar{u}_l''\psi_1+\frac{i}{\alpha R}(\psi_1^{\text{iv}}-2\alpha^2\psi_1''+\alpha^4\psi_1) = 0, \tag{2.1.1}$$

namely the Orr–Sommerfeld equation determining the eigenfunction ψ_1, which is what we would expect as the coefficient of A. Hence (2.23) with $n = 1$ serves to determine the second highest-order term in ϕ_1. It is readily seen that this term is expected to be of the form $A^2\tilde{A}\psi_{11}(z)$, provided that dA/dt is of the form

$$\frac{dA}{dt} = -i\alpha c A + a_1 A^2\tilde{A} + \text{smaller-order terms},$$

where a_1 is some constant. It can be shown that the assumption of a solution in the form

$$\left.\begin{aligned} \phi_1 &= A\psi_1 + A^2\tilde{A}\psi_{11} + \text{smaller-order terms},\\ \phi_2 &= A^2\psi_2 + \text{smaller-order terms},\\ \bar{u} &= \bar{u}_l + A\tilde{A}f_1 + \text{smaller-order terms},\\ \frac{dA}{dt} &= a_0 A + a_1 A^2\tilde{A} + \text{smaller-order terms} \quad (a_0 = -i\alpha c) \end{aligned}\right\} \tag{2.1.2}$$

leads to no inconsistency. In fact further investigation of (2.22) and (2.23) suggests that we look for a solution $\bar{u}$, ϕ_1, ϕ_2, ..., of the form

$$\phi_n = A^n\left\{\psi_n + \sum_{m=1}^{\infty} |A|^{2m}\psi_{mm}\right\} \quad (n \geq 1), \tag{2.1.3}$$

$$\bar{u} = \bar{u}_l + \sum_{m=1}^{\infty} |A|^{2m} f_m, \tag{2.1.4}$$

with

$$\frac{dA}{dt} = A\sum_{m=0}^{\infty} a_m |A|^{2m} \quad (a_0 = -i\alpha c), \tag{2.1.5}$$

where a_m ($m \geqslant 1$) are unknown constants, which also leads to no inconsistency, as we shall see. The above arguments are related to, and are a generalization of, those given by Stuart (1960), who looked for a solution of the same form as (2.1.3), (2.1.4) and (2.1.5), but retained only those terms necessary to give the first approximation to the solutions.

A result needed is the differential equation satisfied by $|A|$, and this is now given. The conjugate of (2.1.5) is

$$\frac{d\tilde{A}}{dt} = \tilde{A} \sum_{m=0}^{\infty} \tilde{a}_m |A|^{2m}, \tag{2.1.6}$$

and the equation for the amplitude of A follows by multiplying (2.1.5) by $\tilde{A}$, (2.1.6) by A and adding the results, giving

$$\frac{d|A|^2}{dt} = |A|^2 \sum_{m=0}^{\infty} (a_m + \tilde{a}_m)|A|^{2m} = 2|A|^2 \sum_{m=0}^{\infty} a_{mr}|A|^{2m}, \tag{2.1.7}$$

where $a_m = a_{mr} + ia_{mi}$. It is evident that $|A|^2$ is monotonic in time between these values which are zeros of the right-hand side of (2.1.7); we shall be interested primarily in the range of $|A|^2$ between zero and the first non-zero positive root of the right-hand side of (2.1.7).

Returning to (2.23), it is readily shown that, on making the substitution (2.1.3) and (2.1.4) and using (2.1.5) and (2.1.7), the left-hand side may be written in the form

$$\begin{aligned}\mathscr{L}(n\alpha)\phi_n \equiv A^n\Bigg[&L(n\alpha)\left\{\psi_n + \sum_{m=1}^{\infty}|A|^{2m}\psi_{nm}\right\} \\ &+\left(c - \frac{i}{\alpha}\sum_{m=0}^{\infty} a_m|A|^{2m} + \sum_{m=1}^{\infty}|A|^{2m}f_m\right) \\ &\qquad\times\left\{(\psi''_n - n^2\alpha^2\psi_n) + \sum_{m=1}^{\infty}|A|^{2m}(\psi''_{nm} - n^2\alpha^2\psi_{nm})\right\} \\ &-\frac{2i}{n\alpha}\left(\sum_{m=0}^{\infty}a_{mr}|A|^{2m}\right)\sum_{m=1}^{\infty}m|A|^{2m}(\psi''_{nm} - n^2\alpha^2\psi_{mn}) - \left(\sum_{m=1}^{\infty}|A|^{2m}f''_m\right) \\ &\qquad\times\left(\psi_n + \sum_{m=1}^{\infty}|A|^{2m}\psi_{nm}\right)\Bigg],\end{aligned} \tag{2.1.8}$$

where
$$L(n\alpha) \equiv (\bar{u}_l - c)\left(\frac{\partial^2}{\partial z^2} - n^2\alpha^2\right) - \bar{u}''_l + \frac{i}{n\alpha R}\left(\frac{\partial^2}{\partial z^2} - n^2\alpha^2\right)^2. \tag{2.1.9}$$

It is easily seen that the right-hand side of (2.23) also contains A^n as a factor. Hence, dividing both sides of (2.23) by A^n, rearranging terms and using the fact that $a_0 = -i\alpha c$, we find that (2.23) becomes

$$\begin{aligned}L(n\alpha)&\left\{\psi_n + \sum_{m=1}^{\infty}|A|^{2m}\psi_{nm}\right\} - \frac{2ic_i}{n}\sum_{m=1}^{\infty}m|A|^{2m}(\psi''_{nm} - n^2\alpha^2\psi_{nm}) \\ &= \frac{i}{\alpha}\left(\sum_{m=1}^{\infty}a_m|A|^{2m}\right)(\psi''_n - n^2\alpha^2\psi_n) + \frac{1}{n}\Bigg[\sum_{m=1}^{n-1}m\psi_m(\psi'''_{n-m} - (n-m)^2\alpha^2\psi'_{n-m}) \\ &\quad - \sum_{m=1}^{n-1}(n-m)\psi'_m(\psi''_{n-m} - (n-m)^2\alpha^2\psi_{n-m})\Bigg] + \sum_{m=1}^{\infty}|A|^{2m}g_{nm}(z) \quad (n \geqslant 1),\end{aligned} \tag{2.1.10}$$

where the summations from $m = 1$ to $m = n-1$ are omitted when $n = 1$, and where g_{nm} is a definite known function† of the ψ's, f's and a's appearing in (2.1.3) to (2.1.5). Now (2.1.10) must be true at all times for which the solution con-

† The function g_{nm} is given explicitly by Watson (1959).

verges and so for all $|A|$ sufficiently small. Hence the coefficients of like powers of $|A|^2$ in (2.1.10) must cancel out. Similarly, on using (2.1.7), the equation of mean motion (2.22) becomes

$$\frac{1}{R}\left(\bar{u}_l'' + \sum_{m=1}^{\infty} |A|^{2m} f_m''\right) - 2\left(\sum_{m=0}^{\infty} a_{mr}|A|^{2m}\right)\left(\sum_{m=1}^{\infty} m|A|^{2m} f_m\right)$$
$$= p^* + i\alpha \sum_{n=1}^{\infty} n|A|^{2n}\left\{\left(\psi_n' + \sum_{m=1}^{\infty}|A|^{2m}\psi_{nm}'\right)\left(\tilde{\psi}_n + \sum_{m=1}^{\infty}|A|^{2m}\tilde{\psi}_{nm}\right)\right.$$
$$\left. - \left(\psi_n + \sum_{m=1}^{\infty}|A|^{2m}\psi_{nm}\right)\left(\tilde{\psi}_n' + \sum_{m=1}^{\infty}|A|^{2m}\tilde{\psi}_{nm}'\right)\right\}', \qquad (2.1.11)$$

in which the coefficients of like powers of $|A|^2$ must cancel as in (2.1.10). It follows from this equation that p^*, a function of time only, must be of the form

$$p^* = \sum_{n=0}^{\infty} k_n |A|^{2n},$$

in which the k's are real and $k_0 = \bar{u}_l''/R = -2/R$. The constants k_n $(n \geqslant 1)$ are completely arbitrary and so must be specified in some way. The simplest case is chosen, namely, that k_n $(n \geqslant 1)$ is zero, which corresponds to the condition that the mean pressure gradient in the flow direction does not change with time.†

Equating the terms independent of $|A|$ in (2.1.10), we obtain the equation

$$L(n\alpha)\psi_n = \frac{1}{n}\left[\sum_{m=1}^{n-1} m\psi_m(\psi_{n-m}''' - (n-m)^2\alpha^2\psi_{n-m}')\right.$$
$$\left. - \sum_{m=1}^{n-1}(n-m)\psi_m'(\psi_{n-m}'' - (n-m)^2\alpha^2\psi_{n-m})\right], \qquad (2.1.12)$$

which becomes the Orr–Sommerfeld equation, $L(\alpha)\psi_1 = 0$, for $n = 1$, as anticipated. The terms independent of $|A|$ in (2.1.11) already cancel. The balancing of the coefficients of $|A|^2$ in (2.1.11) leads to the equation

$$\frac{1}{R}f_1'' - 2\alpha c_i f_1 = i\alpha(\psi_1'\tilde{\psi}_1 - \psi_1\tilde{\psi}_1')' \qquad (2.1.13)$$

on using the result $a_{0r} = \alpha c_i$. Equating coefficients of $|A|^2$ in (2.1.10) and using the result $a_0 = -i\alpha c$, we get

$$L(n\alpha)\psi_{n1} - \frac{2ic_i}{n}(\psi_{n1}'' - n^2\alpha^2\psi_{n1})$$
$$= \frac{ia_1}{\alpha}(\psi_n'' - n^2\alpha^2\psi_n) + g_{n1}(z)$$
$$= \left(\frac{ia_1}{\alpha} - f_1\right)(\psi_n'' - n^2\alpha^2\psi_n) + f_1''\psi_n + \frac{1}{n}\left[\psi_{n+1}'(\tilde{\psi}_1'' - \alpha^2\tilde{\psi}_1) + (n+1)\psi_{n+1}(\tilde{\psi}_1''' - \alpha^2\tilde{\psi}_1')\right.$$
$$-(n+1)\tilde{\psi}_1'(\psi_{n+1}'' - (n+1)^2\alpha^2\psi_{n+1}) - \tilde{\psi}_1(\psi_{n+1}''' - (n+1)^2\alpha^2\psi_{n+1}')$$
$$-\sum_{m=1}^{n-1}(n-m)\{\psi_m'(\psi_{n-m1}'' - (n-m)^2\alpha^2\psi_{n-m1}) + \psi_{m1}'(\psi_{n-m}'' - (n-m)^2\alpha^2\psi_{n-m})\}$$
$$\left. + \sum_{m=1}^{n-1} m\{\psi_m(\psi_{n-m1}''' - (n-m)^2\alpha^2\psi_{n-m1}') + \psi_{m1}(\psi_{n-m}''' - (n-m)^2\alpha^2\psi_{n-m}')\}\right], \qquad (2.1.14)$$

† An alternative condition to this is that the mass flux does not change with time. This condition reduces to $\int_{-1}^{1} f_m\,dz = 0$ which determines the k_n's and involves trivial modifications to the function f_m.

where again it is understood that the summations from $m = 1$ to $m = n-1$ are to be omitted when $n = 1$. Furthermore, from the coefficients of $|A|^{2m}$ $(m > 1)$ in (2.1.11), we obtain the equation

$$\begin{aligned}
&\frac{1}{R} f''_m - 2m\alpha c_i f_m \\
&\quad = 2\sum_{p=1}^{m-1} (m-p)\, a_{pr} f_{m-p} + i\alpha \Big[m(\psi'_m \tilde{\psi}_m - \psi_m \tilde{\psi}'_m)' \\
&\qquad + (m-1)\,(\psi'_{m-1}\tilde{\psi}_{m-11} + \psi'_{m-11}\tilde{\psi}_{m-1} - \tilde{\psi}'_{m-1}\psi_{m-11} - \tilde{\psi}'_{m-11}\psi_{m-1})' \\
&\qquad + \sum_{n=1}^{m-2} n \Big\{ \psi'_n \tilde{\psi}_{nm-n} + \psi'_{nm-n}\tilde{\psi}_n \\
&\qquad + \sum_{p=1}^{m-n-1} \psi'_{np}\tilde{\psi}_{nm-n-p} - \tilde{\psi}'_n \psi_{nm-n} - \tilde{\psi}'_{nm-n}\psi_n - \sum_{p=1}^{m-n-1} \tilde{\psi}'_{np}\psi_{nm-n-p} \Big\}' \Big],
\end{aligned} \tag{2.1.15}$$

where the summation from $n = 1$ to $n = m-2$ is to be omitted for $m = 2$. Also, from the coefficients of $|A|^{2m}$ $(m > 1)$ in (2.1.10), we obtain the equation

$$L(n\alpha)\,\psi_{nm} - \frac{2mic_i}{n}(\psi''_{nm} - n^2\alpha^2\psi_{nm}) = \frac{ia_m}{\alpha}(\psi''_n - n^2\alpha^2\psi_n) + g_{nm}(z), \tag{2.1.16}$$

where g_{nm} is a definite function of the ψ's, f's and a's, and will be a known function of z on reaching the stage when this equation is to be solved.

Having obtained the differential equations satisfied by the functions of z in (2.1.3) and (2.1.4), we have still to state the boundary conditions. These follow from (2.12) which must be satisfied at all times, so that from (2.1.3) and (2.1.4) the boundary conditions are

$$\psi_n, \quad \psi_{nm}, \quad \psi'_n, \quad \psi'_{nm}, \quad f_m = 0 \quad \text{at} \quad z = \pm 1 \quad (n = 1, 2, \ldots;\ m = 1, 2, \ldots). \tag{2.1.17}$$

Moreover, since ψ_1 is an even function of z, it is readily seen from (2.1.12) and (2.1.17) that ψ_2 is odd, ψ_3 is even and, in general, that ψ_n is even for $n = 1, 3, 5, \ldots$, and ψ_n is odd for $n = 2, 4, 6, \ldots$. Again with the homogeneous conditions (2.1.17) it is seen from (2.1.13) that f_1 is even. Similarly, it can be shown from (2.1.14), that ψ_{11} is even and, in general, ψ_{n1} is even for odd values of n and odd for even n. By repeating this argument with equations (2.1.15) and (2.1.16), it can be shown that (i) ψ_n, ψ_{nm} are even functions of z for odd values of n and odd functions of z for n even, and (ii) f_m are even functions of z. It is thus sufficient to solve the problem in, say, the upper half of the channel, $0 \leqslant z \leqslant 1$, in which case the boundary conditions (2.1.17) become

$$\left.\begin{aligned}
\psi_n, \psi_{nm}, \psi'_n, \psi'_{nm}, f_m &= 0 \quad \text{at} \quad z = 1 \quad (n = 1, 2, \ldots;\ m = 1, 2, \ldots), \\
\psi'_n, \psi'_{nm}, \psi'''_n, \psi'''_{nm}, f'_m &= 0 \quad \text{at} \quad z = 0 \quad (n = 1, 3, 5, \ldots;\ m = 1, 2, \ldots), \\
\psi_n, \psi_{nm}, \psi''_n, \psi''_{nm} &= 0 \quad \text{at} \quad z = 0 \quad (n = 2, 4, 6, \ldots;\ m = 1, 2, \ldots).
\end{aligned}\right\} \tag{2.1.18}$$

With these boundary conditions it is hoped to determine ψ_n, f_1, ψ_{n1} together with a_1 from (2.1.12), (2.1.13) and (2.1.14), and f_m, ψ_{nm} together with a_m $(m > 1)$ from (2.1.15) and (2.1.16).

From the homogeneous equation (2.1.1), together with the corresponding boundary conditions in (2.1.18), the eigenfunction, ψ_1, is found to within an arbitrary multiple. In order to make ψ_1 definite we follow Stuart in selecting that ψ_1 for which $\psi_1(0) = 1$. Since the right-hand side of (2.1.12) with $n = 2$ is a function of ψ_1 only, then this fourth-order differential equation for ψ_2 together with the associated boundary conditions in (2.1.18) determine ψ_2. When ψ_2 is determined, the function ψ_3 can be determined and in general since the right-hand side of (2.1.12) is a function of $\psi_1, \psi_2, \ldots, \psi_{n-1}$ only, this equation for ψ_n, together with the boundary conditions in (2.1.18), serve to determine ψ_n. In other words ψ_2, ψ_3, etc., can be determined successively. In a similar manner f_1 is determined from (2.1.13) and the boundary conditions given in (2.1.18). The eigenfunction ψ_1 is identical with Stuart's (1960), while ψ_2 and f_1 are the same as Stuart's ψ_2 and f respectively to the order he considered.

The next step is to solve (2.1.14) with $n = 1$ subject to the corresponding boundary conditions in (2.1.18). At this stage the right-hand side of this equation is known apart from the constant a_1. This fourth-order differential equation with the four boundary conditions leads to a solution which is linear in a_1. By some means a_1, and correspondingly ψ_{11}, must be determined. To this end let us first consider the case in which the function A is $Ce^{-i\alpha ct}$. In such a case, $a_1 = a_2 = \ldots = 0$, and the equation for ψ_{11} becomes

$$L(\alpha)\,\psi_{11} - 2ic_i(\psi''_{11} - \alpha^2\psi_{11}) = g_{11}. \tag{2.1.19}$$

We are interested in a convergent solution for which the function A or $|A|$ is small for all time, and we hope to find such a solution near to the neutral curve $c_i = 0$. Thus c_i will be small, and this fact is used to solve (2.1.19) by expanding in c_i. On the left-hand side of (2.1.19) the second term will then be small compared with the first term, which is composed of the Orr–Sommerfeld operator acting upon ψ_{11}. Hence we may choose four independent parts of the complementary function, one of which will almost satisfy the boundary conditions, that is, it will almost be the eigenfunction ψ_1. It follows from this that the highest-order term in ψ_{11} is probably a multiple of ψ_1 and moreoever that the multiple will tend to infinity as $c_i \to 0$. Accordingly, we look for a solution of the form

$$\psi_{11} = c_i^{-p}\,\psi_{11}^{(-p)} + c_i^{-p+1}\,\psi_{11}^{(-p+1)} + \ldots, \tag{2.1.20}$$

where p is a positive integer and $\psi_{11}^{(-p)}, \psi_{11}^{(-p+1)}, \ldots$ are bounded as $c_i \to 0$. Now it is most likely that $p = 1$, and we shall consider this case. Other cases will be dealt with later. The equations to be satisfied by the $\psi_{11}^{(r)}$ are

$$L(\alpha)\,\psi_{11}^{(-1)} = 0, \tag{2.1.21}$$

$$L(\alpha)\,\psi_{11}^{(0)} = g_{11} + 2i(\psi_{11}^{(-1)''} - \alpha^2\psi_{11}^{(-1)}), \tag{2.1.22}$$

$$\left.\begin{aligned} L(\alpha)\,\psi_{11}^{(1)} &= 2i(\psi_{11}^{(0)''} - \alpha^2\psi_{11}^{(0)}),\\ \ldots \qquad & \quad \ldots\\ \ldots \qquad & \quad \ldots \end{aligned}\right\} \tag{2.1.23}$$

in which $\psi_{11}^{(-1)}, \psi_{11}^{(0)}, \ldots$ all satisfy the same boundary conditions as ψ_{11}. The solution of (2.1.21) is $\psi_{11}^{(-1)} = \lambda\psi_1$, where λ is an arbitrary constant. On substituting this value into the right-hand side of (2.1.22) we obtain a fourth-order differential equation containing the arbitrary constant λ. If we define χ_2, χ_3, χ_4

to be the solutions of the Orr–Sommerfeld equation satisfying the boundary conditions

$$\left.\begin{aligned} \chi_2 = 0, \quad \chi_2' = 1, \quad \chi_2'' = \chi_2''' = 0, \\ \chi_3 = \chi_3' = 0, \quad \chi_3'' = 1, \quad \chi_3''' = 0, \\ \chi_4 = \chi_4' = \chi_4'' = 0, \quad \chi_4''' = 1, \end{aligned}\right\} \quad \text{at} \quad z = 0, \tag{2.1.24}$$

then ψ_1, χ_2, χ_3, χ_4 are four linearly independent solutions of the Orr–Sommerfeld equation, of which ψ_1, χ_3 are even functions of z and χ_2, χ_4 are odd. It will be necessary to calculate χ_3, but we do not need to know χ_2 or χ_4. Following Stuart (1960), we first of all determine λ by multiplying (2.1.22) by the known function

$$\Phi = (\chi_3'' - \alpha^2\chi_3) - (\chi_3''(1) - \alpha^2\chi_3(1))\,(\psi_1'' - \alpha^2\psi_1)/\psi_1''(1)$$

(which satisfies the equation adjoint to the Orr–Sommerfeld equation; it also satisfies the same boundary conditions† as ψ_1) and integrating with respect to z between 0 and 1. This gives

$$\lambda = -\int_0^1 \Phi g_{11}\,dz/2i\int_0^1 \Phi(\psi_1'' - \alpha^2\psi_1)\,dz.$$

Having determined λ, the right-hand side of (2.1.22) is now a completely known even function, and since the solution of (2.1.22) is to be an even function, it must have the form

$$\psi_{11}^{(0)} = A\psi_1 + B\chi_3 + P,$$

where P is any even particular integral of (2.1.22). Either of the two conditions at the wall will determine B and the other condition at the wall will be satisfied. This gives $\psi_{11}^{(0)}$ apart from the arbitrary constant A. Next the value of $\psi_{11}^{(0)}$ is substituted into the first equation of (2.1.23), which will then contain the unknown constant A. This constant is then determined in exactly the same way as λ was determined and $\psi_{11}^{(1)}$ is found to within an arbitrary constant multiple of ψ_1 by the procedure just described. In this way ψ_{11} can be determined completely.

The solution obtained when $A(t)$ is chosen to be proportional to $e^{-i\alpha ct}$ converges for only a very small interval in time whereas we wish to find a solution which converges for all time. The first step in finding such a solution is to choose a suitable value for a_1. The solution of (2.1.14) with $n = 1$ subject to the corresponding boundary conditions in (2.1.18) consists of the sum of the solution of (2.1.19) and the solution of

$$L(\alpha)\,\psi_{11} - 2ic_i(\psi_{11}'' - \alpha^2\psi_{11}) = \frac{ia_1}{\alpha}(\psi_1'' - \alpha^2\psi_1) \tag{2.1.25}$$

subject to the same boundary conditions, namely, $-(a_1/2\alpha c_i)\,\psi_1$. Since a_1 is arbitrary, ψ_{11} is determined to within an arbitrary additive multiple of ψ_1.‡

† That $\Phi'(1) = 0$ can be seen by integrating by parts the relation

$$\int_0^1 (\chi_3'' - \alpha^2\chi_3)\,L(\alpha)\,\psi_1 dz = 0$$

and making use of the identities, $L(\alpha)\chi_3 \equiv \{L(\alpha)\chi_3\}'' \equiv 0$.

‡ It can be shown that, if to any given ψ_{11} is added a multiple of ψ_1 then, by redefining the functions A, ψ_{nm} and f_m ($m \geqslant 2$) and rearranging the series (2.1.3), (2.1.4) and (2.1.5), a similar series in the new functions is obtained in which ψ_{11}, ψ_n, f_1 are the same as the original functions. In other words the arbitrariness in ψ_{11} corresponds to the arbitrariness in which the series may be rearranged in this way. The same property occurs in all the functions ψ_{1m}.

By choosing a value of a_1 such that $(\lambda/c_i)-(a_1/2\alpha c_i)$ is bounded as $c_i \to 0$, we shall obtain a function ψ_{11} which is also bounded as $c_i \to 0$. Any such value of a_1 will suffice; in general there is no further restriction on a_1 which will make the series converge more rapidly. We may in fact choose $a_1 = 2\alpha\lambda$. The solution in this case is readily seen to be given by (2.1.20) with $p = 1$ apart from the first term; that is, the solution is

$$\psi_{11} = \psi_{11}^{(0)} + c_i \psi_{11}^{(1)} + \ldots, \tag{2.1.26}$$

where $\psi_{11}^{(0)}$, $\psi_{11}^{(1)}$, etc., are found from (2.1.22) and (2.1.23) in which $\psi_{11}^{(-1)}$ is to be replaced by $\lambda\psi_1$. Having determined a_1 and ψ_{11}, the ψ_{n1} follow successively from (2.1.14), and f_2 follows from (2.1.15) subject to the boundary conditions in (2.1.18).

At this stage g_{12} is a known function of z. It can be shown that when $a_2 = 0$, ψ_{12} can be expressed as in (2.1.20) with $p = 1$, where the $\psi_{12}^{(r)}$ are bounded as $c_i \to 0$. It also follows that this function ψ_{12} will consist of the sum of an unbounded multiple of ψ_1 and a bounded function and, by exactly the same reasoning as given for a_1 and ψ_{11}, a_2 can be chosen to make ψ_{12} bounded and a_2 will also be bounded. Then ψ_{n2} follow successively (g_{n2} being a known function of z when the equation for ψ_{n2} is to be solved) and f_3, after which ψ_{13} and a_3 are similarly determined. We cannot proceed in this way indefinitely for, in (2.1.16) with $n = 1$, although c_i is assumed to be very small, ultimately m is so large that mc_i is no longer small and we can no longer apply the reasoning given immediately after (2.1.19). However, in such a case we can solve for ψ_{1m} directly without any trouble; when mc_i is very small, solving for ψ_{1m} directly leads to ill-conditioned algebraic equations and we have to resort to expanding as in (2.1.20), but when mc_i is not small there is no longer any reason why the algebraic equations should be ill-conditioned. Hence when mc_i is not small, in general no particular value of a_m will make the series converge more rapidly than any other value, and so we can specify a_m to be zero. Proceeding in this way, all the functions of z and all the constants a_m in (2.1.3), (2.1.4) and (2.1.5) can be found. It should be noted that the degree of convergence of the perturbation series depends upon obtaining sufficiently good approximations for the first terms in the expansions. This in turn depends upon mc_i being very small compared with the width of the critical layer where the functions vary rapidly. As the width of the critical layer is of order $(\alpha R)^{-\frac{1}{3}}$, we therefore cannot expect to obtain solutions by means of a perturbation series unless $m\,|c_i|\,(\alpha R)^{\frac{1}{3}} \ll 1$. Furthermore, we cannot expect to obtain an improvement on linearized theory unless the perturbation series can be applied to ψ_{11}; this means that we insist on c_i being so small that $|c_i|\,(\alpha R)^{\frac{1}{3}} \ll 1$. It then appears that the perturbation series is to be applied as long as $m\,|c_i|\,(\alpha R)^{\frac{1}{3}} \ll 1$, but when m is so large that this is not true then a_m is specified to be zero and the resulting equations are to be solved directly.

If a_{1r} tends to a non-zero constant as $c_i \to 0$, then a first approximation to $|A|$ is obtained by retaining only the first two terms on the right-hand side of (2.1.7). This first approximation is therefore given by

$$|A|^2 = \frac{\alpha c_i K \exp{[2\alpha c_i t]}}{(1 - a_{1r} K \exp{[2\alpha c_i t]})}, \tag{2.1.27}$$

in which K is a real arbitrary constant. As $|A| \to 0$, that is, as $c_i t \to -\infty$, $|A|^2 \sim \alpha c_i K \exp[2\alpha c_i t]$. But since $A \sim C e^{-i\alpha ct}$ as $A \to 0$ then we can say that C and K are related by

$$\alpha c_i K = |C|^2. \tag{2.1.28}$$

The equilibrium amplitude $|A|_e$ is found, to the first approximation, from (2.1.27) to be

$$|A|_e^2 = -\frac{\alpha c_i}{a_{1r}}. \tag{2.1.29}$$

If a_{1r} turns out to be positive, then this equilibrium amplitude can be found in the subcritical case ($c_i < 0$), while if a_{1r} is negative the equilibrium amplitude can be found in the supercritical case ($c_i > 0$). The equilibrium amplitude is independent of K, which is not surprising since the arbitrariness of K or $|C|^2$ merely reflects the arbitrariness of the origin of time and in any equilibrium state $|A|_e$ will be independent of time. There is therefore no loss in generality if we assume that $|C|^2 = \alpha\,|c_i|$, so that $K = \operatorname{sgn} c_i$ and (2.1.27) becomes

$$|A|^2 = \frac{\alpha\,|c_i| \exp[2\alpha c_i t]}{(1 - a_{1r} \operatorname{sgn} c_i \exp[2\alpha c_i t])}. \tag{2.1.30}$$

Since $|A|$ is of order $|c_i|^{\frac{1}{2}}$ and all the functions of z in (2.1.3), (2.1.4) are bounded then the solution should converge for small enough values of c_i, that is, near enough to the neutral curve.

To obtain $|A|$ without approximation, we first of all denote $|A|^2$ by y and change the independent variable in (2.1.7) to $x = \exp[2\alpha c_i t]$. Equation (2.1.7) then becomes

$$\alpha c_i x y' = y(\alpha c_i + a_{1r} y + a_{2r} y^2 + \dots), \tag{2.1.31}$$

a dash denoting differentiation with respect to x. From the condition that A behaves like $C e^{-i\alpha ct}$ as $A \to 0$ it follows that $|A|^2 \exp(-2\alpha c_i t) \to |C|^2 = \alpha\,|c_i|$ as $c_i t \to -\infty$, that is, that $y/x \to \alpha\,|c_i|$ as $x \to 0$. We look for a solution of the form

$$y = \alpha c_i y_1 + \alpha^2 c_i^2 y^2 + \dots, \tag{2.1.32}$$

where $y_1, y_2, \dots$ satisfy the differential equations found on formally equating equal powers of αc_i. These equations are

$$\left.\begin{aligned} x y_1' &= y_1 + a_{1r} y_1^2, \\ x y_n' &= y_n + 2a_{1r} y_1 y_n + F_n(y_1, y_2, \dots, y_{n-1}) \quad (n \geqslant 2), \end{aligned}\right\} \tag{2.1.33}$$

and the boundary conditions to be satisfied are

$$\frac{y_1}{x} \to \operatorname{sgn} c_i, \quad \frac{y_n}{x} \to 0 \quad (n \geqslant 2) \quad \text{as} \quad x \to 0.$$

The solutions of (2.1.33) which satisfy these boundary conditions are

$$\left.\begin{aligned} y_1 &= \frac{(\operatorname{sgn} c_i)\,x}{[1 - a_{1r}(\operatorname{sgn} c_i)\,x]}, \\ y_n &= \frac{x}{[1 - a_{1r}(\operatorname{sgn} c_i)\,x]^2} \int_0^x \frac{[1 - a_{1r}(\operatorname{sgn} c_i)\,x]^2 F_n(y_1, y_2, \dots, y_{n-1})\,dx}{x^2}, \end{aligned}\right\} \tag{2.1.34}$$

giving the full solution for $|A|^2$ from (2.1.32). It may happen that a_{1r} tends to zero as $c_i \to 0$ in which case $|A|^2$ would be of order $|c_i|^{\frac{1}{2}}$ at least; $|A|^2$ could be found using an analysis similar to that just given.

The argument of A follows from (2.1.5) and (2.1.6) by multiplying the first by $1/\tilde{A}$, the second by $-A/\tilde{A}^2$ and adding the resulting equations. Integration of the resulting equation yields

$$A/\tilde{A} = D\exp(-2i\alpha c_r t)\exp\left\{\frac{i}{\alpha c_i}\int_0^x\left(\sum_{m=1}^{\infty} a_{mi}y^m\right)\frac{dx}{x}\right\}, \tag{2.1.35}$$

where D is a constant and x, y have the same meanings as in the previous paragraph. But since $A\,e^{i\alpha ct} \to C$ as $A \to 0$ then $A/\tilde{A} \to \exp(-2i\alpha c_r t)\,C/\tilde{C}$ as $A \to 0$. Letting $x \to 0$ in (2.1.35) we deduce that $D = C/\tilde{C}$ so that (2.1.35) may be written in the form

$$A/\tilde{A} = C/\tilde{C}\exp[-2i\alpha\mathscr{C}t], \tag{2.1.36}$$

where

$$\mathscr{C} = c_r - \frac{1}{\alpha\log x}\int_0^x\left(\sum_{m=1}^{\infty} a_{mi}y^m\right)\frac{dx}{x}$$

is the velocity of the wave and it changes with time. As $x \to 0$, that is, as $A \to 0$, $\mathscr{C} \to c_r$ as it should; while

$$\mathscr{C} \to c_r - \frac{1}{\alpha}\sum_{m=1}^{\infty} a_{mi}|A|_e^{2m}$$

as $x \to \infty$, the value of the wave velocity in the state of equilibrium. When an equilibrium state exists, as in the case we are considering, then the same analysis shows that there are similar disturbances for $|A| > |A|_e$, although probably a second equilibrium state could not be calculated. If the equilibrium states exist in the supercritical region ($c_i > 0$), then these disturbances decay with time to the state of equilibrium so that the state of equilibrium is a state of stable equilibrium to disturbances of this kind. On the other hand, if the equilibrium states exist in the subcritical region ($c_i < 0$), then these disturbances amplify with time, so that the state of equilibrium is a state of unstable equilibrium, which shows that although the flow is stable to infinitesimal disturbances in this region it can be unstable to certain finite disturbances. Note that, because a_m is zero for m large enough, the series in (2.1.36) and (2.1.31) are polynomials in y.

This analysis shows how the functions ϕ_n, $\bar{u}$ given by (2.1.3) and (2.1.4) are determined, after which u', w' may be found from (2.9) and (2.10). The part $p^{**}(z,t)$ of the pressure in (2.11) is then found to within an arbitrary additive function of time from (2.17) as $p^{**} = -\overline{w'^2}$ when p^{**} is expressed as a series in $|A|^2$ with coefficients as functions of z and coefficients of equal powers of $|A|^2$ are equated. Similarly, the functions $p_n(z,t)$ in (2.11) are found from (2.18) on equating components of $e^{ni\alpha x}$, expressing $p_n(z,t)$ as the product of A^n with a series in $|A|^2$ whose coefficients are functions of z, dividing the equation for p_n by the factor A^n and equating the coefficients of equal powers of $|A|^2$. Equation (2.19) will then be satisfied automatically.

We have been considering the most likely case, namely that in which the solution of (2.1.19) has the form (2.1.20) with $p = 1$. It is possible, however, that we may find, in the notation used above, that $\lambda \to \infty$ as $c_i \to 0$. Then the most probable situation is that in which the form of the solution is given by

(2.1.20) with $p = 2$, in which $\psi_{11}^{(-2)}$ is proportional to ψ_1. In this case the most probable situation is when the equations to be solved are

$$L(\alpha)\,\psi_{11}^{(-2)} = 0, \tag{2.1.37}$$

$$L(\alpha)\,\psi_{11}^{(-1)} = 2i(\psi_{11}^{(-2)''} - \alpha^2\psi_{11}^{(-2)}), \tag{2.1.38}$$

$$L(\alpha)\,\psi_{11}^{(0)} = g_{11} + 2i(\psi_{11}^{(-1)''} - \alpha^2\psi_{11}^{(-1)}), \tag{2.1.39}$$

$$\left.\begin{array}{l} L(\alpha)\,\psi_{11}^{(1)} = 2i(\psi_{11}^{(0)''} - \alpha^2\psi_{11}^{(0)}), \\ \cdots \qquad \cdots \\ \cdots \qquad \cdots \end{array}\right\} \tag{2.1.40}$$

The solution of (2.1.37) is $\psi_{11}^{(-2)} = \mu\psi_1$, where μ is an arbitrary constant. Then (2.1.38) becomes

$$L(\alpha)\,\psi_{11}^{(-1)} = 2i\mu(\psi_1'' - \alpha^2\psi_1). \tag{2.1.41}$$

The situation which is being considered is when there exists a solution P (an even function of z) of

$$L(\alpha)\,\psi_{11}^{(-1)} = 2i(\psi_1'' - \alpha^2\psi_1), \tag{2.1.42}$$

which satisfies the boundary conditions. This situation is most probable in the sense that, if α is regarded as fixed, then ψ_1, c_r, R most probably are regular functions of c_i, when ψ_{11} has a double pole in c_i. In such a case it can be shown that there does exist a solution of (2.1.42) which satisfies the boundary conditions. Note that λ is infinite in this case, for it is readily shown, by multiplying (2.1.42) by Φ and integrating with respect to z from 0 to 1, that

$$\int_0^1 \Phi(\psi_1'' - \alpha^2\psi_1)\,dz = 0. \tag{2.1.43}$$

It follows from (2.1.42) that

$$\psi_{11}^{(-1)} = A\psi_1 + \mu P.$$

Then (2.1.39) becomes

$$L(\alpha)\,\psi_{11}^{(0)} = g_{11} + 2iA(\psi_1'' - \alpha^2\psi_1) + 2i\mu(P'' - \alpha^2 P). \tag{2.1.44}$$

The constant μ is now determined in exactly the same way as λ was determined above; equation (2.1.44) is multiplied by Φ and integrated with respect to z between 0 and 1, and, after making use of (2.1.43), we obtain

$$\mu = -\int_0^1 \Phi g_{11}\,dz \Big/ 2i\int_0^1 \Phi(P'' - \alpha^2 P)\,dz.$$

Having determined μ, if P_1 is any (even) particular integral of (2.1.44) with the term in A omitted then, since $\psi_{11}^{(0)}$ is to be even, it must have the form

$$\psi_{11}^{(0)} = A_1\psi_1 + B_1\chi_3 + AP + P_1.$$

Either of the two boundary conditions at the wall will determine B_1, and the other condition at the wall will automatically be satisfied. This gives $\psi_{11}^{(0)}$ apart from the constants A_1, A. The constant A is determined from the next equation in the same way as μ has just been determined. Proceeding in the same manner, the required solutions of the equations (2.1.37) to (2.1.40) can be found, giving the full solution of (2.1.19).

The solution of (2.1.14) is the sum of the solution which we have just found and the function $-(a_1/2\alpha c_i)\,\psi_1$. The constant a_1 is chosen in order to make $c_i\psi_{11}$ bounded as $c_i \to 0$. Since the highest-order terms in ψ_{11} are given by $\{(\mu/c_i^2) - (a_1/2\alpha c_i)\}\,\psi_1$, then $c_i\psi_{11}$ will be bounded if we choose $a_1 = 2\alpha\mu/c_i$. With this choice of a_1, both $c_i\psi_{11}$ and $c_i a_1$ are bounded while ψ_{11} and a_1 become infinite as $c_i \to 0$. The function ψ_{11} is given by (2.1.20) with $p = 1$, where $\psi_{11}^{(-1)}$, $\psi_{11}^{(0)}$ are the solutions of (2.1.38) with $\psi_{11}^{(-2)}$ replaced by $\mu\psi_1$, (2.1.39) and (2.1.40) which satisfy the boundary conditions. The functions ψ_{n1}, f_2 are found from (2.1.14), (2.1.15) after expressing them in series similar to that for ψ_{11}; both $c_i\psi_{n1}$ and $c_i f_2$ are bounded, and ψ_{n1}, f_2 become infinite as $c_i \to 0$. From (2.1.16), by using the same arguments as we used for ψ_{11}, the function ψ_{12} is expressed as a series similar to (2.1.20) with $p = 4$, the highest-order term being a multiple of ψ_1, and, by suitable choice of a_2 of order $1/c_i^3$, ψ_{12} becomes a function of the same order. The functions ψ_{n2} and f_3 are also of order $1/c_i^3$. Similarly, ψ_{13} is of order c_i^{-5} by suitable choice of a_3 (of order c_i^{-5}), etc. For the same reason as given above, we insist on c_i being so small that $|c_i|\,(\alpha R)^{\frac{1}{3}} \ll 1$. It then appears that the perturbation series is to be applied so long as $m\,|c_i|\,(\alpha R)^{\frac{1}{3}} \ll 1$, but when m is so large that this is not true then a_m is specified to be zero and the resulting equations are to be solved directly. The series on the right-hand side of (2.1.7) has been reduced to a polynomial in $|A|^2$. If there is no positive zero of this polynomial, then no equilibrium state can be found although the solution may converge more rapidly than the linearized solution and so be valid for larger amplitudes. On the other hand, if there is a positive zero of this polynomial, then the smallest of these will give a definite value of $|A|_e^2$ of order c_i^2, and the analysis applies to both the subcritical and supercritical cases. In this case, in (2.1.3), apart from the first term, the first few terms will be of the same order in c_i. The convergence of the series will depend on how ψ_{nm}, f_m behave for large values of m (a very rough order of magnitude analysis suggests that ψ_n, ψ_{nm} behave satisfactorily for large n, if m is regarded as fixed).

This last remark applies to all the theory presented here. Even if the most probable situation occurs (when $p = 1$ in (2.1.20)), there is no guarantee that the series will converge, or even represent a solution asymptotically as $c_i \to 0$, as t becomes large. However, one would expect the theory to be an improvement over linearized theory for a range in time, and, moreover, it does seem likely that, if the most probable case arises, the series will either converge or represent a solution asymptotically.

2.2. *Solution for Couette flow*

The analysis for the Couette flow problem is so similar to that for the Poiseuille flow problem that only a short sketch of the method will be given. Here the highest-order term in ϕ_1 is assumed to be of the form $A(t)\,\psi_1(z) + \tilde{A}(t)\,\tilde{\psi}_1(-z)$. By similar arguments to those used in § 2.1, a solution is sought in which

$$\left.\begin{aligned}
\phi_1 &= \{A\psi_1(z) + \tilde{A}\tilde{\psi}_1(-z)\} + \{A^3\psi_{11}^{(1)}(z) + A^2\tilde{A}\psi_{11}^{(2)}(z) + A\tilde{A}^2\psi_{11}^{(3)}(z) + \tilde{A}^3\psi_{11}^{(4)}(z)\}\\
&\quad + \text{smaller-order terms},\\
\phi_2 &= \{A^2\psi_2^{(1)}(z) + A\tilde{A}\psi_2^{(2)}(z) + \tilde{A}^2\psi_2^{(3)}(z)\} + \text{smaller-order terms},\\
\bar{u} &= \bar{u}_l + \{A^2 f_1^{(1)}(z) + A\tilde{A} f_1^{(2)}(z) + \tilde{A}^2 f_1^{(3)}(z)\} + \text{smaller-order terms},\\
\frac{dA}{dt} &= a_0 A + (a_1^{(1)}A^3 + a_1^{(2)}A^2\tilde{A} + a_1^{(3)}A\tilde{A}^2 + a_1^{(4)}\tilde{A}^3) + \text{smaller-order terms}\\
&\quad (a_0 = -i\alpha c).
\end{aligned}\right\} \quad (2.2.1)$$

It is obvious from this and (2.22) and (2.23) what the form of the full expansion is corresponding to (2.1.3), (2.1.4) and (2.1.5). When (2.2.1) is substituted into (2.22) and (2.23), the terms are collected into those of the same order in $|A|$. Since, for $|A|$ sufficiently small, $|A|$ varies monotonically with time, then the terms of the same order in $|A|$ must balance. Applying this to the highest-order terms in (2.23) with n = 1, we deduce that

$$AL(z, \alpha)\,\psi_1 - \tilde{A}\tilde{L}(-z,\alpha)\,\tilde{\psi}_1(-z) = 0, \tag{2.2.2}$$

where $L(z, \alpha)$ is the Orr-Sommerfeld operator (2.1.9) with $n = 1$ and we have used the fact that $\bar{u}_l$ is an odd function of z. Now (2.2.2) can be rewritten in the form

$$\frac{A}{\tilde{A}}L(z,\alpha)\,\psi_1 - \tilde{L}(-z,\alpha)\,\tilde{\psi}_1(-z) = 0,$$

in which $A/\tilde{A}$ varies with time. In order that this equation be satisfied, it follows that

$$L(z,\alpha)\,\psi_1 = 0,\ \tilde{L}(-z,\alpha)\,\tilde{\psi}_1(-z) = 0, \text{ with } \psi_1, \psi_1' = 0 \text{ at } z = \pm 1. \tag{2.2.3}$$

The second equation is satisfied automatically when the first one is satisfied, so that (2.2.3) is equivalent to the single equation

$$L(\alpha)\,\psi_1 = 0 \text{ with } \psi_1 = \psi_1' = 0 \text{ at } z = \pm 1, \tag{2.2.4}$$

which is the Orr-Sommerfeld equation for the determination of the eigenfunction ψ_1. In (2.23) with n = 2, the balancing of the highest-order terms leads to an equation of the form

$$A^2\chi_1(z) + A\tilde{A}\chi_2(z) + \tilde{A}^2\chi_3(z) = 0, \tag{2.2.5}$$

or

$$(A/\tilde{A})^2\chi_1 + (A/\tilde{A})\,\chi_2 + \chi_3 = 0,$$

which can only be satisfied if $\chi_1 = \chi_2 = \chi_3 = 0$. In other words, the coefficients of $A^2, A\tilde{A}, \tilde{A}^2$ vanish separately. The equation $\chi_1 = 0$ is in fact

$$L(2\alpha)\,\psi_2^{(1)} = -\tfrac{1}{2}(\psi_1'\psi_1'' - \psi_1\psi_1''') \tag{2.2.6}$$

with the boundary conditions $\psi_2^{(1)} = \psi_2^{(1)\prime} = 0$ at $z = \pm 1$.

For the moment let $\tilde{\psi}_1(-z)$ *be replaced by* $\psi_2(z)$; then $\chi_2 = 0$ yields

$$L(2\alpha)\,\psi_2^{(2)} + c_r(\psi_2^{(2)\prime\prime} - 4\alpha^2\psi_2^{(2)}) = -\tfrac{1}{2}(\psi_1'\psi_2'' + \psi_2'\psi_1'' - \psi_1\psi_2''' - \psi_2\psi_1'''), \tag{2.2.7}$$

with the boundary conditions $\psi_2^{(2)} = \psi_2^{(2)\prime} = 0$ at $z = \pm 1$. If the sign of z is changed in the equation $\chi_3 = 0$, then the complex conjugate of the resulting equation is (2.2.6) with $\psi_2^{(1)}(z)$ replaced by $\tilde{\psi}_2^{(3)}(-z)$, and the boundary conditions become $\tilde{\psi}_2^{(3)}(-z) = \tilde{\psi}_2^{(3)\prime}(-z) = 0$ at $z = \pm 1$, so that $\tilde{\psi}_2^{(3)}(-z) \equiv \psi_2^{(1)}(z)$ or

$$\psi_2^{(3)}(z) \equiv \tilde{\psi}_2^{(1)}(\text{ -2}). \tag{2.2.8}$$

Equations (2.2.6), (2.2.7) and (2.2.8) determine $\psi_2^{(1)}, \psi_2^{(2)}, \psi_2^{(3)}$.

The highest-order terms in (2.22) give an identity since p^* is zero and $\bar{u}_l = z$. The second highest-order terms in (2.22) give an equation of the form (2.2.5), so that again $\chi_1 = \chi_2 = \chi_3 = 0$. The equation $\chi_1 = 0$ is

$$\frac{1}{R}f_1^{(1)\prime\prime} + 2i\alpha c f_1^{(1)} = i\alpha(\psi_1'\tilde{\psi}_2 - \psi_1\tilde{\psi}_2')', \tag{2.2.9}$$

with the boundary conditions $f_1^{(1)} = 0$ at $z = \pm 1$. Since $\psi_2(z) \equiv \tilde{\psi}_1(-z)$, the right-hand side of (2.2.9) is an odd function of z, so that $f_1^{(1)}$ is also an odd function of z. Hence we need only consider the range $0 \leqslant z \leqslant 1$ with the boundary conditions $f_1^{(1)} = 0$ at $z = 0, 1$. The equation $\chi_2 = 0$ is

$$\frac{1}{R} f_1^{(2)''} - 2\alpha c_i f_1^{(2)} = i\alpha(\psi_1' \tilde{\psi}_1 - \psi_1 \tilde{\psi}_1' + \psi_2' \tilde{\psi}_2 - \psi_2 \tilde{\psi}_2')' \tag{2.2.10}$$

with the boundary conditions $f_1^{(2)} = 0$ at $z = \pm 1$. The right-hand side of (2.2.10) is a real odd function of z, and hence so is $f_1^{(2)}$. Thus we need only consider the range $0 \leqslant z \leqslant 1$ with the boundary conditions $f_1^{(2)} = 0$ at $z = 0, 1$. The complex conjugate of the equation $\chi_3 = 0$ is the equation (2.2.9) with $f_1^{(1)}(z)$ replaced by $f_1^{(3)}(z)$, so that

$$f_1^{(3)}(z) \equiv \tilde{f}_1^{(1)}(z). \tag{2.2.11}$$

Equations (2.2.9), (2.2.10) and (2.2.11) determine $f_1^{(1)}, f_1^{(2)}, f_1^{(3)}$.

The balancing of the second highest-order terms in (2.23) with $n = 1$ leads to an equation of the form

$$A^3\chi_1(z) + A^2\tilde{A}\chi_2(z) + A\tilde{A}^2\chi_3(z) + \tilde{A}^3\chi_4(z) = 0, \tag{2.2.12}$$

from which $\chi_1 = \chi_2 = \chi_3 = \chi_4 = 0$ as before. The equation $\chi_1 = 0$ [where it may be permissible to remind the reader that $\psi_2(z) \equiv \psi_1(-z)$] is

$$\begin{aligned} L(\alpha)\,\psi_{11}^{(1)} - 2c(\psi_{11}^{(1)''} - \alpha^2\psi_{11}^{(1)}) = {} & \left(\frac{ia_1^{(1)}}{\alpha} - f_1^{(1)}\right)(\psi_1'' - \alpha^2\psi_1) + f_1^{(1)''}\psi_1 \\ & + \frac{i\tilde{a}_1^{(4)}}{\alpha}(\psi_2'' - \alpha^2\psi_2) + \psi_2^{(1)'}(\tilde{\psi}_2'' - \alpha^2\tilde{\psi}_2) + 2\psi_2^{(1)}(\tilde{\psi}_2''' - \alpha^2\tilde{\psi}_2') \\ & - 2\tilde{\psi}_2'(\psi_2^{(1)''} - 4\alpha^2\psi_2^{(1)}) - \tilde{\psi}_2(\psi_2^{(1)'''} - 4\alpha^2\psi_2^{(1)}), \end{aligned} \tag{2.2.13}$$

with $\psi_{11}^{(1)} = \psi_{11}^{(1)'} = 0$ at $z = \pm 1$. The equation $\chi_2 = 0$ is

$$\begin{aligned} L(\alpha)\,\psi_{11}^{(2)} - 2ic_i(\psi_{11}^{(2)''} - \alpha^2\psi_{11}^{(2)}) = {} & \left(\frac{ia_1^{(2)}}{\alpha} - f_1^{(2)}\right)(\psi_1'' - \alpha^2\psi_1) + f_1^{(2)''}\psi_1 \\ & + \left(\frac{i\tilde{a}_1^{(3)}}{\alpha} - f_1^{(1)}\right)(\psi_2'' - \alpha^2\psi_2) + f_1^{(1)''}\psi_2 + \psi_2^{(1)'}(\tilde{\psi}_1'' - \alpha^2\tilde{\psi}_1) + 2\psi_2^{(1)}(\tilde{\psi}_1''' - \alpha^2\tilde{\psi}_1') \\ & - 2\tilde{\psi}_1'(\psi_2^{(1)''} - 4\alpha^2\psi_2^{(1)}) - \tilde{\psi}_1(\psi_2^{(1)'''} - 4\alpha^2\psi_2^{(1)'}) + \psi_2^{(2)'}(\tilde{\psi}_2'' - \alpha^2\tilde{\psi}_2) \\ & + 2\psi_2^{(2)}(\tilde{\psi}_2''' - \alpha^2\tilde{\psi}_2') - 2\tilde{\psi}_2'(\psi_2^{(2)''} - 4\alpha^2\psi_2^{(2)}) - \tilde{\psi}_2(\psi_2^{(2)'''} - 4\alpha^2\psi_2^{(2)'}), \end{aligned} \tag{2.2.14}$$

with $\psi_{11}^{(2)} = \psi_{11}^{(2)'} = 0$ at $z = \pm 1$. Together with (2.2.13) and (2.2.14) there are two similar equations $\chi_3 = 0$ and $\chi_4 = 0$. Now the solution of (2.2.13) is linear in both $a_1^{(1)}$ and $\tilde{a}_1^{(4)}$, the coefficient of $a_1^{(1)}$ being $-(i/2\alpha c)\,\psi_1$ and the coefficient of $\tilde{a}_1^{(4)}$ being $-\{i/\alpha(3c+\tilde{c})\}\,\tilde{\psi}_1(-z)$, both of which satisfy the boundary conditions. We are interested in values of α and R which correspond to very small values of c_i. In general c_r will not be small, so that the solution of (2.2.13) with $a_1^{(1)}$ and $a_1^{(4)}$ put equal to zero will not become infinite as $c_i \to 0$. Accordingly, no particular values of $a_1^{(1)}$ and $a_1^{(4)}$ will make the solution more convergent than any other pair of values. Let us then specify that $a_1^{(1)}$ and $a_1^{(4)}$ be zero, so that $\psi_{11}^{(1)}$ is determined from (2.2.13) with $a_1^{(1)}$ and $a_1^{(4)}$ equal to zero. Similarly the solution of (2.2.14) is linear in both $a_1^{(2)}$ and $\tilde{a}_1^{(3)}$, the coefficient of $a_1^{(2)}$ being $-(1/2\alpha c_i)\,\psi_1$ and the coefficient of $\tilde{a}_1^{(3)}$ being $-(i/2\alpha c)\,\psi_2$. As in the calculation of ψ_{11} in § 2.1, the solution of (2.2.14) with $a_1^{(2)}$ and $a_1^{(3)}$ put equal to zero will become infinite as $c_i \to 0$, the highest-order term in the solution being proportional to $c_i^{-1}\psi_1$. No value of $\tilde{a}_1^{(3)}$ will result in making $\psi_{11}^{(2)}$ bounded so that we shall specify $a_1^{(3)}$ to be zero. Then $a_1^{(2)}$ plays exactly the same role in the calculation of $\psi_{11}^{(2)}$ as a_1 did in the calculation of ψ_{11} in § 2.1; $a_1^{(2)}$ is chosen to reduce the order of magnitude of $\psi_{11}^{(2)}$ as $c_i \to 0$ by exactly the same method. Note that the equation for A is of the same form as the equation for A in § 2.1 up to terms of order $|A|^3$. From the equations $\chi_3 = 0$, $\chi_4 = 0$ we obtain

$$\left.\begin{aligned} \psi_{11}^{(3)}(z) &\equiv \tilde{\psi}_{11}^{(2)}(-z), \\ \text{and} \qquad \psi_{11}^{(4)}(z) &\equiv \tilde{\psi}_{11}^{(1)}(-z), \end{aligned}\right\} \tag{2.2.15}$$

respectively.

The method which has just been described can be used to find smaller-order terms and from further investigation it is evident that we can look for a solution of the form

$$\left.\begin{aligned} \phi_1 &= (A\psi_1(z) + \tilde{A}\tilde{\psi}_1(-z)) + (A^3\psi_{11}^{(1)}(z) + A^2\tilde{A}\psi_{11}^{(2)}(z) + A\tilde{A}^2\tilde{\psi}_{11}^{(2)}(-z) \\ &\quad + \tilde{A}^3\tilde{\psi}_{11}^{(1)}(-z)) + \ldots, \\ \phi_2 &= (A^2\psi_2^{(1)}(z) + A\tilde{A}\psi_2^{(2)}(z) + \tilde{A}^2\tilde{\psi}_2^{(1)}(-z)) + \ldots, \\ u &= \bar{u}_l + (A^2 f_1^{(1)}(z) + A\tilde{A} f_1^{(2)}(z) + \tilde{A}^2\tilde{f}_1^{(1)}(z)) + \ldots, \\ \frac{dA}{dt} &= A\sum_{m=0}^{\infty} a_m |A|^{2m} \quad (a_0 = -i\alpha c), \end{aligned}\right\} \tag{2.2.16}$$

where we have replaced $a_1^{(2)}$ by a_1. The functions ψ_1, $\psi_2^{(1)}$, $\psi_2^{(2)}$, $f_1^{(1)}$, $f_1^{(2)}$, $\psi_{11}^{(1)}$, $\psi_{11}^{(2)}$ and a_1 are found from (2.2.4), (2.2.6), (2.2.7), (2.2.9), (2.2.10) and (2.2.13) with $a_1^{(1)}$ and $a_1^{(4)}$ zero, (2.2.14) with $a_1^{(3)}$ zero and $a_1^{(2)}$ replaced by a_1. The function A has exactly the same form and is calculated in exactly the same way as in § 2.1. Also as in § 2.1, α and R must be chosen so that the corresponding values of c_i and $c_i(\alpha R)^{\frac{1}{3}}$ are sufficiently small for the method to apply. In Couette flow, which is believed to be stable to infinitesimal disturbances, there might be some difficulty in finding values of α and R which satisfy these conditions (whereas in Poiseuille flow, c_i can be made arbitrarily small, more or less independently of $(\alpha R)^{\frac{1}{3}}$, by choice of a point (α, R) sufficiently close to the neutral curve). Moreover, this problem will correspond only to the subcritical problem of Poiseuille flow so that, even if suitable values of α and R are found which satisfy the above conditions, in order to obtain a worthwhile solution a_{1r} must turn out to be positive in the general case.

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